

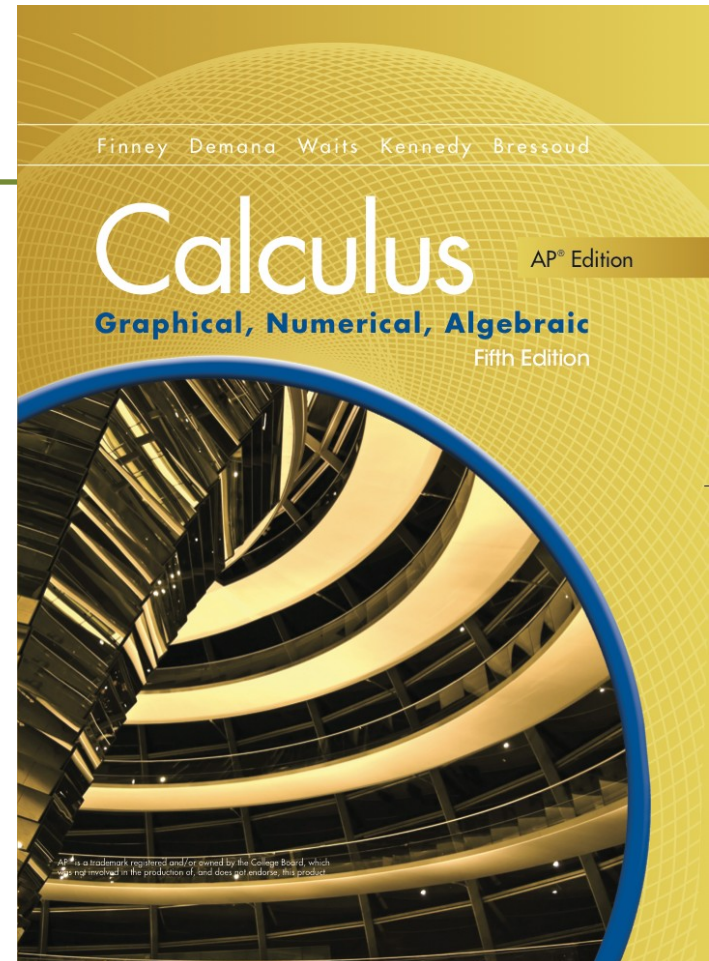
Chapter 8

Applications of Definite Integral



Section 8.3

Volumes



Quick Review

Give a formula for the area of the plane region in terms of the single variable x

1. a square with side length x
2. a semicircle of radius x
3. a semicircle of diameter x
4. an equilateral triangle with sides of length x
5. an isosceles triangle with two sides of length $2x$ and one of length x

Quick Review Solutions

Give a formula for the area of the plane region in terms of the single variable x

1. a square with side length x x^2
2. a semicircle of radius x $\pi x^2 / 2$
3. a semicircle of diameter x $\pi x^2 / 8$
4. an equilateral triangle with sides of length x $(\sqrt{3}/4) x^2$
5. an isosceles triangle with two sides of length $2x$ and one of length x $(\sqrt{15}/4) x^2$

What you'll learn about

- Volumes as limits of Riemann sums
- Volumes with circular, square, or other cross sections
- Volumes of solids of revolution using washers or cylindrical shells

...and why

The techniques of this section allow us to compute volumes of certain solids in three dimensions.

Volume of a Solid

The definition of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

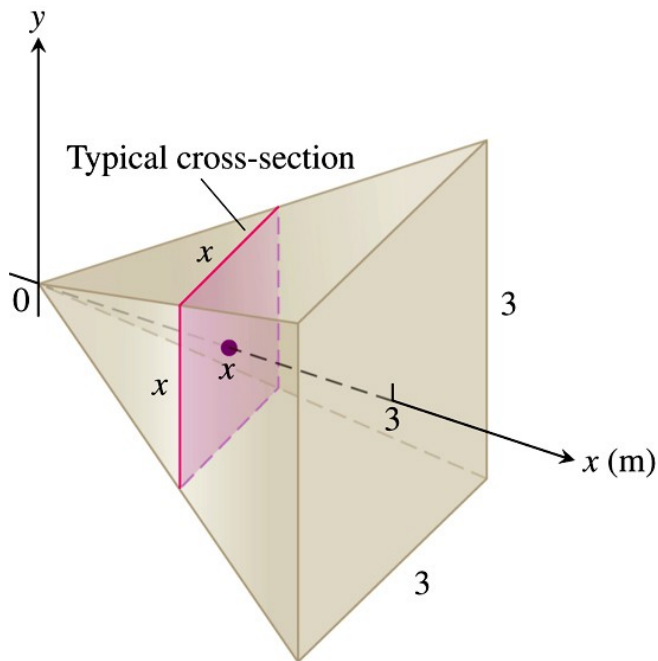
$$V = \int_a^b A(x) dx$$

How to Find Volumes by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

Example Square Cross Sections

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.



1. Sketch: Draw the pyramid with its vertex at the origin and its altitude along the interval $0 \leq x \leq 3$. Sketch a typical cross section at a point x between 0 and 3.

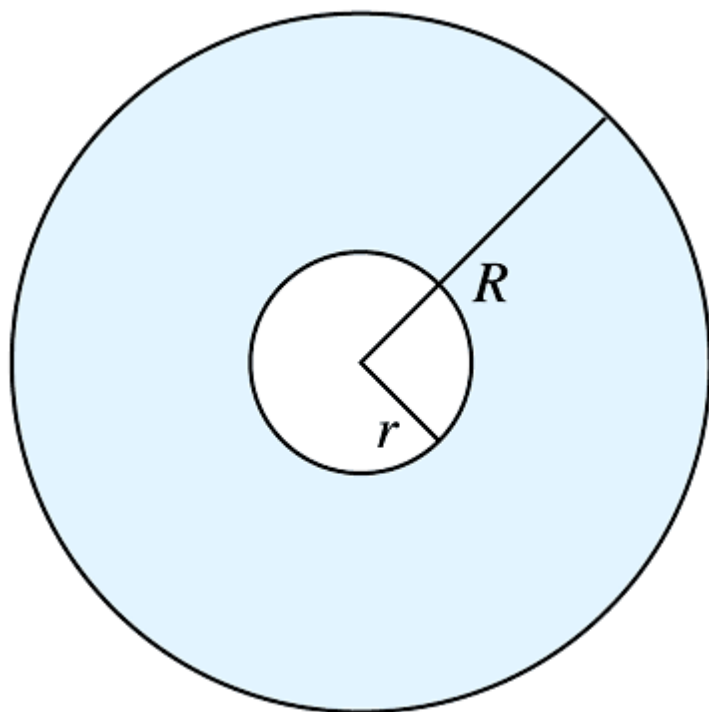
Example Square Cross Sections

2. Find a formula for $A(x)$: The cross section at x is a square x meters on a side, so $A(x) = x^2$.
3. Find the limits of integration: The square goes from $x = 0$ to $x = 3$.
4. Integrate to find the volume:

$$\begin{aligned} V &= \int_0^3 A(x) dx \\ &= \int_0^3 x^2 dx \\ &= \left. \frac{x^3}{3} \right|_0^3 \\ &= 9 \text{ m}^3 \end{aligned}$$

Example A Solid of Revolution

The region between the graph of $f(x) = 2 + x \cos x$ and the x -axis over the interval $[-2, 2]$ is revolved about the x -axis to generate a solid. Find the volume of the solid.



Revolving the region about the x -axis generates a vase-shaped solid. The cross section at a typical point x is circular, with radius equal to $f(x)$.

Its area is $A(x) = \pi (f(x))^2$.

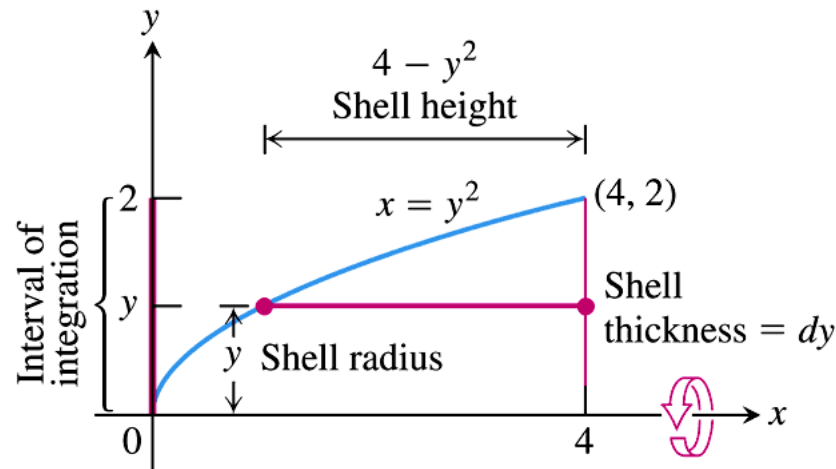
The volume of the solid is

$$\begin{aligned} V &= \int_{-2}^2 A(x) dx \\ &= \text{NINT}(\pi (2 + x \cos x)^2, x, -2, 2) \\ &\approx 52.43 \text{ units cubed.} \end{aligned}$$

Example Finding Volumes Using Cylindrical Shells

The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

1. Sketch the region and draw a line segment across it parallel to the axis of revolution. Label the segment's length and distance from the axis of revolution. The width of the segment is the shell thickness.



Example Finding Volumes Using Cylindrical Shells

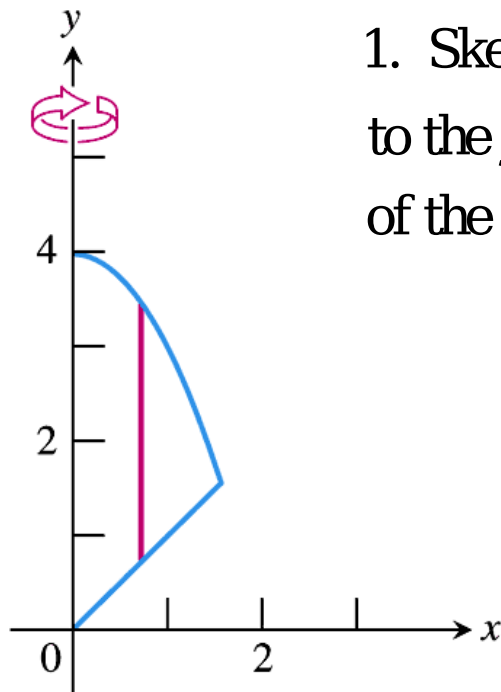
2. Identify the limits of integration: y runs from 0 to 2.

3. Integrate to find the volume:

$$\begin{aligned} V &= \int_0^2 2\pi (\text{shell radius}) (\text{shell height}) \, dy \\ &= \int_0^2 2\pi (y) (4 - y^2) \, dy \\ &= 8\pi \end{aligned}$$

Example Finding Volumes Using Cylindrical Shells

The region bounded by the curve $y = 4 - x^2$, $y = x$, and $x = 0$ is revolved about the y -axis to form a solid. Use cylindrical shells to find the volume of the solid.



1. Sketch the region and draw a line segment across it parallel to the y -axis. The segment's length is $4 - x^2 - x$. The distance of the segment from the axis of revolution is x .

Example Finding Volumes Using Cylindrical Shells

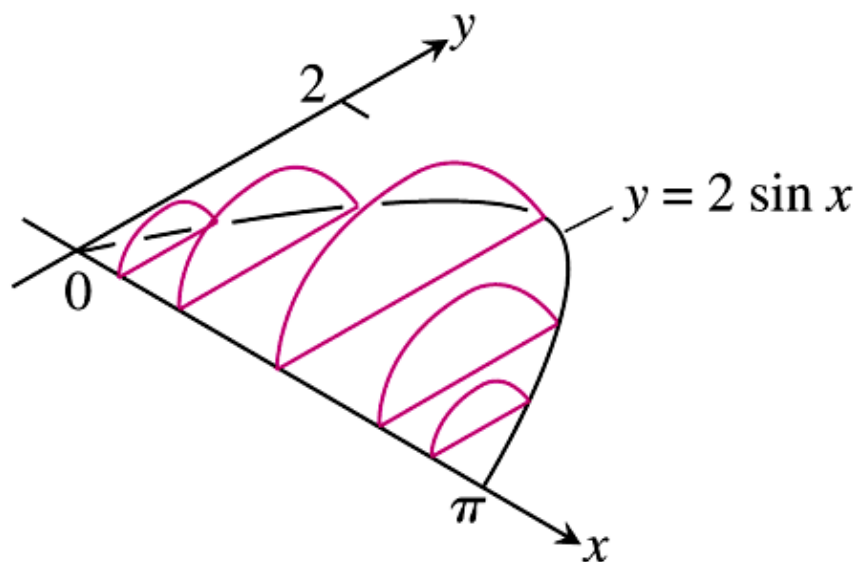
2. Identify the limits of integration: The x -coordinate of the point of intersection of the curves $y = 4 - x^2$ and $y = x$ in the first quadrant is about 1.562. So x runs from 0 to 1.562.

3. Integrate to find the volume:

$$\begin{aligned} V &= \int_0^{1.562} 2\pi (\text{shell radius}) (\text{shell height}) \, dx \\ &= \int_0^{1.562} 2\pi (x) (4 - x^2 - x) \, dx \\ &\approx 13.327 \end{aligned}$$

Example Other Cross Sections

A solid is made so that its base is the shape of the region between the x -axis and one arch of the curve $y = 2\sin x$. Each cross section cut perpendicular to the x -axis is a semicircle whose diameter runs from the x -axis to the curve. Find the volume of the solid.



Example Other Cross Sections

The semicircle at each point x has radius $= \frac{2\sin x}{2} = \sin x$

and area $A(x) = \frac{1}{2}\pi (\sin x)^2$.

$$\begin{aligned}\text{So, } V &= \frac{\pi}{2} \int_0^{\pi} (\sin x)^2 dx \\ &= \frac{\pi}{2} \text{NINT}((\sin x)^2, x, 0, \pi) \\ &= \frac{\pi}{2} (1.570796327) \\ &= \frac{\pi^2}{4} \text{ in}^3.\end{aligned}$$

Quick Quiz Sections 8.1 – 8.3

You may use a graphing calculator to solve the following problems.

1. The base of a solid is the region in the first quadrant bounded by the x -axis, the graph of $y = \sin^{-1} x$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the x -axis is a square.

What is its volume?

- (A) 0.117
- (B) 0.285
- (C) 0.467
- (D) 0.571
- (E) 1.571

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Quick Quiz Sections 8.1 – 8.3

2. Let R be the region in the first quadrant bounded by the graph of $y = 3x - x^2$ and the x -axis. A solid is generated when R is revolved about the vertical line $x = -1$. Set up, but do not integrate, the definite integral that gives the volume of this solid.

(A) $\int_0^3 2\pi (x+1) (3x - x^2) dx$

(B) $\int_1^3 2\pi (x+1) (3x - x^2) dx$

(C) $\int_0^3 2\pi (x) (3x - x^2) dx$

(D) $\int_0^3 2\pi (3x - x^2) dx$

(E) $\int_0^3 (3x - x^2) dx$

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(A) $\int_0^3 2\pi (x+1)(3x - x^2) dx$

(B) $\int_1^3 2\pi (x+1)(3x - x^2) dx$

(C) $\int_0^3 2\pi (x)(3x - x^2) dx$

(D) $\int_0^3 2\pi (3x - x^2) dx$

(E) $\int_0^3 (3x - x^2) dx$

Quick Quiz Sections 8.1 – 8.3

3. A developing country consumes oil at a rate given by $r(t) = 20e^{0.2t}$ million barrels per year, where t is time measured in years, for $0 \leq t \leq 10$. Which of the following expressions gives the amount of oil consumed by the country during the time interval $0 \leq t \leq 10$?

(A) $r(10)$

(B) $r(10) - r(0)$

(C) $\int_0^{10} r'(t) dt$

(D) $\int_0^{10} r(t) dt$

(E) $10 \cdot r(10)$

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